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RESEARCH ARTICLE

Extinction in common property resource models: an analytically tractable example

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Abstract We discuss an analytically tractable discrete-time dynamic game in which a finite number of players extract a renewable resource. We characterize a symmetric Markov-perfect Nash equilibrium of this game and derive a necessary and sufficient condition under which the resource does not become extinct in equilibrium. This condition requires that the intrinsic growth rate of the resource exceeds a certain threshold value that depends on the number of players and on their time-preference rates.

Keywords Tragedy of the commons · Extinction · Markov-perfect Nash equilibrium

JEL Classification C73 · Q20

1 Introduction

The development of an economic theory of common property resources started with the seminal contribution of Gordon (1954). The "tragedy of the commons" received particular attention, following the publication of the well-known paper by Hardin (1968). Such a situation emerges if a common property resource is inefficiently used due to the missing allocation of property rights, and it constitutes an important example

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of the failure of Adam Smith's invisible hand argument.[1] Inefficient use of a resource may result in its over-exploitation or, even worse, in its extinction. Despite the prominent status of the tragedy of the commons in the economics literature, there are only few treatments in which the precise conditions under which extinction occurs can be identified under realistic assumptions on the resource dynamics. The purpose of this note is to present one more case in which this is possible.

The possibility that extinction of a renewable resource is the result of optimal behavior of rational agents has been recognized in various papers (see, e.g., Lewis and Schmalensee (1977) and Cropper et al. (1979) for early examples). It is known from that literature that the optimality of extinction depends on the relative sizes of the time-preference rate of the agents on the one hand and the intrinsic growth rate of the resource on the other hand (see, e.g., Cropper (1988)). The very same parameters appear also in the condition derived in the present paper, but the number of competing agents turns out to be crucial as well.

The explicit recognition of the interdependence of agents in the context of a dynamic model of fishery through a stock externality was made by Smith (1968). Dasgupta (1981) went one step further by developing a dynamic model of a fishery and analyzing over-exploitation and extinction as possible consequences of the tragedy of the commons. Even if his model has the flavor of a dynamic game, the players do not act as dynamic optimizers. The first explicit dynamic game formulation of the tragedy of the commons is "the great fish war" from Levhari and Mirman (1980).

The model of Levhari and Mirman (1980) features a logarithmic utility function for the players and a Cobb–Douglas natural growth function of the resource, and the authors analyze a Markov-perfect Nash equilibrium of their dynamic game. The slope of the Cobb–Douglas growth function at a vanishing resource stock is infinitely large. This is at odds with the assumptions typically imposed on growth functions of natural resources (see, e.g., Clark (1990)). The present paper therefore generalizes the Levhari–Mirman example by allowing more general production functions with a constant elasticity of substitution (CES). The analytical tractability of this more general framework has already been demonstrated by Benhabib and Rustichini (1994) in a representative agent framework. The key idea that they use is the combination of the CES growth function with a utility function featuring constant relative risk aversion of an appropriate degree. We utilize the same device but exploit it in a multi-player setting.

Dutta and Sundaram (1993) extend the analysis of Levhari and Mirman (1980) and show that under certain conditions, there need not arise a tragedy of the commons in the sense of over-exploitation. As a matter of fact, they construct an example in which a Markov-perfect Nash equilibrium results in the under-exploitation of the resource. Nevertheless, the equilibria studied in Dutta and Sundaram (1993) are always inefficient. Benhabib and Radner (1992), on the other hand, demonstrate the existence of efficient equilibria under the assumption that the players can use history-dependent

[1] Dasgupta (2005) provides a comprehensive account of the early static models which appeared in the economic literature on common property resources. His survey covers some game-theoretic ideas, but these are in static or repeated game frameworks, not in dynamic games such as the one analyzed in the present paper.

trigger strategies. In addition to these (and other) game-theoretic studies of the tragedy of the commons, this problem has also been analyzed in a competitive framework. A notable example is Brander and Taylor (1998), who study the joint dynamics of the resource stock and the population and relate their findings to the history of Easter Island as well as other Polynesian civilizations. Although their model is quite different from ours, the influence of the population size on the eventual fate of the resource stock is in the focus of both studies.

The main contribution of the present paper is the derivation of a very simple condition, which is both necessary and sufficient for asymptotic extinction, that is, for the convergence of the natural resource stock to 0.[2] This condition involves the intrinsic growth rate of the resource at a vanishing population g, the time-preference rate of the agents r, and the number of agents n, and it shows that extinction occurs if the intrinsic growth rate is less than or equal to the time-preference rate multiplied by the number of agents: $g \leq nr$. It is interesting to note that no further properties of the growth function or the preferences of the agents enter the extinction formula.

The paper makes a methodological contribution as well. Whereas Benhabib and Rustichini (1994) restrict themselves to guessing and verifying a particular solution of the Bellman equation, we provide a rigorous proof that this solution is indeed the optimal value function. Due to the fact that the utility function is unbounded, standard theorems like those presented in Stokey and Lucas (1989) are not applicable. Recently, the adaptation of dynamic programming methods to the case of unbounded returns has received considerable attention. The focus of that literature is on deriving sufficient conditions for the Bellman equation to have a unique continuous solution (see, e.g., Rincón-Zapatero and Rodriguez-Palmero (2003, 2009) or Matkowski and Nowak (2011)). In contrast to this literature, we use duality methods to verify that the solution of the Bellman equation is indeed the optimal value function.

The rest of the present paper is organized as follows. In Sect. 2, we review the results from Benhabib and Rustichini (1994) on the representative agent model and provide a rigorous proof (based on duality arguments) that this solution is indeed the optimal value function. In Sect. 3, we then turn to the dynamic game and characterize a symmetric Markov-perfect Nash equilibrium. Using this characterization, we derive the precise condition that leads to extinction. Section 4 concludes the paper by outlining how the results can be extended to a stochastic setting.

2 The representative agent problem

Let us denote by $x_t \geq 0$ the resource stock available at the beginning of period $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, by $c_t \geq 0$ the amount of the resource extracted in period t, and by $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ the natural growth function of the resource, where $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuously differentiable, strictly increasing, and strictly concave function satisfying $f(0) = 0$. The resource stock evolves according to

[2] The concept of asymptotic extinction has to be contrasted with extinction in finite time, which does not occur in the model under consideration.

$$x_{t+1} = f(x_t) - c_t, \tag{1}$$

whereby the initial stock level $x_0 \geq 0$ is given. Suppose that an agent seeks to maximize

$$\sum_{t=0}^{+\infty} \rho^t u(c_t) \tag{2}$$

subject to (1) and the constraints

$$c_t \geq 0 \text{ and } x_{t+1} \geq 0. \tag{3}$$

Here, $\rho \in (0, 1)$ is a time-preference factor, and $u : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$ is a strictly increasing and strictly concave utility function with $u(c) > -\infty$ for all $c > 0$.

Apart from the interpretation as a model of optimal resource exploitation, the above model can also be interpreted in the framework of neoclassical optimal growth theory. It is well known that this model is analytically solvable if the growth function is of the Cobb–Douglas form $f(x) = ax^\alpha$ (with $a > 0$ and $\alpha \in (0, 1)$) and the utility function is logarithmic, i.e., $u(c) = \ln(c)$. The more general case with a CES production function and a CRRA utility function (with corresponding exponents) has been dealt with by Benhabib and Rustichini (1994). These authors have derived a solution to the Bellman equation and have discussed properties of the corresponding optimal policy function.[3] Since we shall be using the same class of models for a study of the tragedy of the commons, we collect the relevant results in this section.

Let the natural growth function and the utility function be given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ a\left(x^{1-\eta} + b\right)^{1/(1-\eta)} & \text{if } x > 0, \end{cases} \tag{4}$$

and

$$u(c) = \begin{cases} -\infty & \text{if } c = 0, \\ \dfrac{c^{1-\eta}}{1-\eta} & \text{if } c > 0, \end{cases} \tag{5}$$

respectively. Here, the parameter values are assumed to satisfy $a > 0$, $b > 0$, and $\eta > 1$. Note that the case of a Cobb–Douglas production function and a logarithmic utility function can be considered as the limiting case as η approaches 1. Furthermore, note that the natural growth function f satisfies all assumptions stated above: It is continuously differentiable, strictly increasing, and strictly concave on $\mathbb{R}_+$, and it holds that $f(0) = 0$, $f'(0) = a$, and $\lim_{x \to +\infty} f'(x) = 0$. The fact that the function f has finite slope at 0 makes it more suitable as a natural growth function of a renewable resource than the Cobb–Douglas specification often used in the context of economic growth.

[3] The focus of Benhabib and Rustichini (1994) is on the implications of various depreciation schemes which they incorporate using a vintage capital approach.

The dynamic optimization problem described by Eqs. (1)–(5) will be denoted as $P(a)$.[4] The Bellman equation for the dynamic optimization problem $P(a)$ is given by

$$W(x) = \sup\{u(c) + \rho W(f(x) - c) \mid 0 \le c \le f(x)\}. \tag{6}$$

Lemma 1 *Assume that $a^{1-\eta}\rho < 1$ and define*

$$A = \frac{1}{\rho}\left[\left(\frac{a^{\eta-1}}{\rho}\right)^{1/\eta} - 1\right]^{-\eta}, \quad B = \frac{Ab}{(1-\eta)(1-\rho)}, \quad \lambda = 1 - \left(\frac{a^{\eta-1}}{\rho}\right)^{-1/\eta}. \tag{7}$$

It holds that $A > 0$, $B < 0$, and $\lambda \in (0, 1)$. Moreover, the function $W : \mathbb{R}_+ \mapsto \mathbb{R}_+ \cup \{-\infty\}$ defined by

$$W(x) = \begin{cases} -\infty & \text{if } x = 0, \\ \dfrac{Ax^{1-\eta}}{1-\eta} + B & \text{if } x > 0 \end{cases} \tag{8}$$

solves the Bellman Eq. (6) and the supremum on the right-hand side of this equation is attained at $c = \lambda f(x)$.

Proof The assumptions $a^{1-\eta}\rho < 1$, $b > 0$, $\rho \in (0, 1)$, and $\eta > 1$ together with (7) imply immediately that $A > 0$, $B < 0$, and $\lambda \in (0, 1)$.

Let $G(c)$ be the maximand on the right-hand side of (6) with f, u, and W as specified by (4), (5), and (8), respectively. If $x = 0$, then the only feasible choice for c is $c = 0$, and it follows that $G(c) = G(0) = -\infty$. Hence, Eq. (6) holds for $x = 0$. Now suppose that $x > 0$. In this case, the feasible set $[0, f(x)]$ is a non-empty and compact interval and it holds that $\lim_{c\to 0} G(c) = G(0) = -\infty$ and $\lim_{c\to f(x)} G(c) = G(f(x)) = -\infty$. Furthermore, since $A > 0$, it follows that $G(c)$ is strictly concave on $[0, f(x)]$ and continuously differentiable on the interior of this interval. Hence, there must exist a unique maximum on the right-hand side of (6) and this maximum must satisfy the first-order condition for an interior extremum, that is,

$$c^{-\eta} = A\rho[f(x) - c]^{-\eta}.$$

This equation has the solution

$$c = \frac{(A\rho)^{-1/\eta} f(x)}{1 + (A\rho)^{-1/\eta}}, \tag{9}$$

[4] Of course, the problem depends also on the parameters b, η, and ρ. For our purpose, however, it is convenient to focus on the dependence on a.

and for this value of c, it holds that

$$G(c) = \frac{(A\rho)^{-(1-\eta)/\eta} f(x)^{1-\eta}}{(1-\eta)[1+(A\rho)^{-1/\eta}]^{1-\eta}} + \rho\left[\frac{Af(x)^{1-\eta}}{(1-\eta)[1+(A\rho)^{-1/\eta}]^{1-\eta}} + B\right]$$
$$= \frac{A\rho[1+(A\rho)^{-1/\eta}]^{\eta} f(x)^{1-\eta}}{1-\eta} + B\rho$$
$$= A\rho[1+(A\rho)^{-1/\eta}]^{\eta} a^{1-\eta} \frac{x^{1-\eta}}{1-\eta} + \frac{A\rho[1+(A\rho)^{-1/\eta}]^{\eta} a^{1-\eta} b}{1-\eta} + B\rho,$$

where we have used the definition of $f(x)$ from Eq. (4). To prove that the function W defined in (8) satisfies the Bellman equation, it is therefore sufficient to show that

$$A = A\rho[1+(A\rho)^{-1/\eta}]^{\eta} a^{1-\eta}$$

and

$$B = \frac{A\rho[1+(A\rho)^{-1/\eta}]^{\eta} a^{1-\eta} b}{1-\eta} + B\rho.$$

It is straightforward to verify that A and B as defined in (7) satisfy these equations. Finally, by substituting A from (7) into (9), we obtain $c = \lambda f(x)$. This completes the proof. □

Although Lemma 1 identifies W as a solution of the Bellman equation, it neither proves that W is the optimal value function nor proves that $c = \lambda f(x)$ is the optimal extraction rule. Standard results from dynamic programming on the uniqueness of the solutions of the Bellman equation are not applicable in the present case, because the utility function is unbounded (see, e.g., Stokey and Lucas (1989)). More recently, these results have been extended to models with unbounded returns (see Rincón-Zapatero and Rodriguez-Palmero (2003, 2009) and Matkowski and Nowak (2011)). The present model fits, for example, the framework of Matkowski and Nowak (2011) so that their proposition 3 could be applied to verify that W is indeed the optimal value function. In the "Appendix," however, we present an alternative approach that is based on duality arguments.

Now consider the dynamics generated by the optimal consumption rule. These dynamics are described by the difference equation

$$x_{t+1} = h(x_t) := (1-\lambda) f(x_t). \tag{10}$$

We have the following result.

Lemma 2 *Assume that* $a^{1-\eta}\rho < 1$.

(a) *If* $a\rho > 1$, *then there exists a unique positive value* x^* *such that* $h(x^*) = x^*$. *This value is given by* $x^* = [bK/(1-K)]^{1/(1-\eta)}$, *where* $K = (a\rho)^{(1-\eta)/\eta}$. *All trajectories of (10) that start from an initial state* $x_0 > 0$ *converge to* x^*.

(b) *If $a\rho \leq 1$, then there does not exist a positive fixed point of the difference Eq. (10). All trajectories of (10) converge to $x = 0$, whereby it holds that $\lim_{t\to+\infty} \rho^t x_{t+1}^{1-\eta} = 0$.*

Proof We have

$$x_{t+1} = (1-\lambda) f(x_t) = (1-\lambda) a (x_t^{1-\eta} + b)^{1/(1-\eta)},$$

where λ is given in (7). Hence, it follows that

$$x_{t+1}^{1-\eta} = K(x_t^{1-\eta} + b)$$

with K as defined in the statement of the lemma. Thus, denoting $x_t^{1-\eta}$ by y_t, we get the linear difference equation

$$y_{t+1} = K y_t + K b$$

which has the solution

$$y_t = \begin{cases} \left(y_0 - \dfrac{Kb}{1-K}\right) K^t + \dfrac{Kb}{1-K} & \text{if } K \neq 1, \\ y_0 + tb & \text{if } K = 1. \end{cases}$$

In the case $a\rho > 1$, we have $K \in (0, 1)$ and it follows that y_t converges to $bK/(1-K)$ as t approaches $+\infty$. Consequently, $\lim_{t\to+\infty} x_t = x^*$. When $a\rho = 1$, then y_t approaches $+\infty$ as $t \to +\infty$ and it follows that $\lim_{t\to+\infty} x_t = 0$. Moreover, we have

$$\lim_{t\to+\infty} \rho^t x_{t+1}^{1-\eta} = \lim_{t\to+\infty} \rho^t y_{t+1} = \lim_{t\to+\infty} \rho^t [y_0 + (t+1)b] = 0.$$

Finally, when $a\rho < 1$, then we have $K > 1$ and it follows again that y_t diverges to $+\infty$ as $t \to +\infty$ and that $\lim_{t\to+\infty} x_t = 0$. In this case, we have

$$\lim_{t\to+\infty} \rho^t x_{t+1}^{1-\eta} = \lim_{t\to+\infty} \rho^t y_{t+1} = \lim_{t\to+\infty} \rho^t \left[\left(y_0 - \frac{Kb}{1-K}\right) K^{t+1} + \frac{Kb}{1-K}\right] = 0,$$

because $\rho K = (a^{1-\eta}\rho)^{1/\eta} < 1$ holds by our assumption that $a^{1-\eta}\rho < 1$. □

3 The common property resource game

In this section, we assume that there are $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ agents who extract the common resource stock. We denote by $c_{i,t}$ the extraction by agent i in period t. The agents are identical and seek to maximize the individual utility functional

$$\sum_{t=0}^{+\infty} \rho^t u(c_{i,t}) \tag{11}$$

subject to the state Eq. (1) with $c_t = \sum_{i=1}^n c_{i,t}$ and subject to the constraints

$$c_{i,t} \geq 0 \text{ and } x_{t+1} \geq 0. \tag{12}$$

We assume that the agents act non-cooperatively. A strategy for agent i is a function $\sigma^{(i)} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ that maps the current resource stock x_t to the agent's consumption rate $c_{i,t}$, that is, $c_{i,t} = \sigma^{(i)}(x_t)$. The strategy space of agent i is the set of all such functions. A strategy profile is a n-tuple $(\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)})$ consisting of one strategy for each of the n players. A strategy profile $(\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)})$ is feasible, if

$$\sum_{j=1}^{n} \sigma^{(j)}(x) \leq f(x)$$

holds for all $x \in \mathbb{R}_+$. A strategy profile $(\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)})$ is called symmetric if there exists a strategy σ such that $\sigma^{(i)} = \sigma$ holds for all $i \in \{1, 2, \dots, n\}$.

A strict Markov-perfect Nash equilibrium (strict MPNE) is a feasible strategy profile such that for all $i \in \{1, 2, \dots, n\}$, it holds that the problem of maximizing (11) subject to (12) and

$$x_{t+1} = f(x_t) - \sum_{j \neq i} \sigma^{(j)}(x_t) - c_{i,t} \tag{13}$$

has a unique solution and this solution satisfies $c_{i,t} = \sigma^{(i)}(x_t)$ for all $t \in \mathbb{N}_0$.

Theorem 1 *Assume that* $a^{1-\eta}\rho < 1$ *and* $n \in \mathbb{N}$.

(a) *There exists a symmetric MPNE* $(\sigma, \sigma, \dots, \sigma)$ *of the form* $\sigma(x) = \lambda_n f(x)$, *where* $\lambda_n \in (0, 1/n)$.
(b) *It holds that* λ_n *is strictly decreasing with respect to* n *with* $\lambda_1 = \lambda$ *and* $\lim_{n \to +\infty} \lambda_n = 0$.

Proof Suppose that $n - 1$ players choose the strategy $\sigma(x) = \lambda_n f(x)$. The state Eq. (13) for the optimization problem of the remaining player is then given by

$$x_{t+1} = f_n(x) - c_{i,t}$$

where $f_n(x) = a_n(x^{1-\eta} + b)^{1/(1-\eta)}$ and

$$a_n = a[1 - (n-1)\lambda_n]. \tag{14}$$

In the terminology introduced in Sect. 2, the remaining player faces the dynamic optimization problem $P(a_n)$. According to the results from Sect. 2, the unique solution of that problem is given by the extraction rule

$$c_{i,t} = \left[1 - \left(\frac{a_n^{\eta-1}}{\rho}\right)^{-1/\eta}\right] f_n(x_t) = \frac{a_n}{a}\left[1 - \left(\frac{a_n^{\eta-1}}{\rho}\right)^{-1/\eta}\right] f(x_t)$$

provided that the parameter condition

$$a_n^{1-\eta}\rho < 1 \tag{15}$$

holds. Part (a) of the theorem is therefore proven if it can be shown that there exist numbers $a_n > 0$ and $\lambda_n \in (0, 1/n)$ such that (14), (15), and

$$\lambda_n = \frac{a_n}{a}\left[1 - \left(\frac{a_n^{\eta-1}}{\rho}\right)^{-1/\eta}\right] \tag{16}$$

hold simultaneously. Using (16) to eliminate λ_n from (14), we obtain after some rearrangements

$$(n-1)\left[1 - \left(\frac{a_n^{\eta-1}}{\rho}\right)^{-1/\eta}\right] = \frac{a}{a_n} - 1. \tag{17}$$

For $n = 1$, this obviously implies that $a_1 = a$, and using (16), it follows that $\lambda_1 = \lambda$. Hence, we recover the results from Sect. 2.

To deal with the case $n > 1$, let us define

$$L(z, n) = (n-1)\left[1 - \left(\frac{z^{\eta-1}}{\rho}\right)^{-1/\eta}\right]$$

and

$$R(z) = \frac{a}{z} - 1.$$

Note that Eq. (17) can then be written as $L(a_n, n) = R(a_n)$. The following arguments are illustrated by Fig. 1. It is easy to see that both $L(z, n)$ and $R(z)$ are continuous functions of $z \in (0, +\infty)$. The mapping $z \mapsto L(z, n)$ is strictly increasing, whereas the mapping $z \mapsto R(z)$ is strictly decreasing. Furthermore, it holds that $\lim_{z\to 0} L(z, n) = -\infty$, $\lim_{z\to+\infty} L(z, n) = n - 1$, $\lim_{z\to 0} R(z) = +\infty$, and $\lim_{z\to+\infty} R(z) = -1$. This proves that for every $n > 1$, there exists a unique value $z^*(n)$ that satisfies $L(z^*(n), n) = R(z^*(n))$. Define $a_n = z^*(n)$ for each $n > 1$. Then, a_n is the unique solution to (17) for each $n > 1$.

As $a^{1-\eta}\rho < 1$, it follows that $L(a, n) > 0$ and $R(a) = 0$. This shows that the solution of (17) must satisfy $a_n < a$ (see Fig. 1 again). Note that this implies that $L(a_n, n) = R(a_n) > R(a) = 0$. The property $L(a_n, n) > 0$, in turn, implies that condition (15) is satisfied. Finally, we obtain from $n > 1$, (15), and (17) that

$$\frac{a}{a_n} - 1 < n - 1.$$

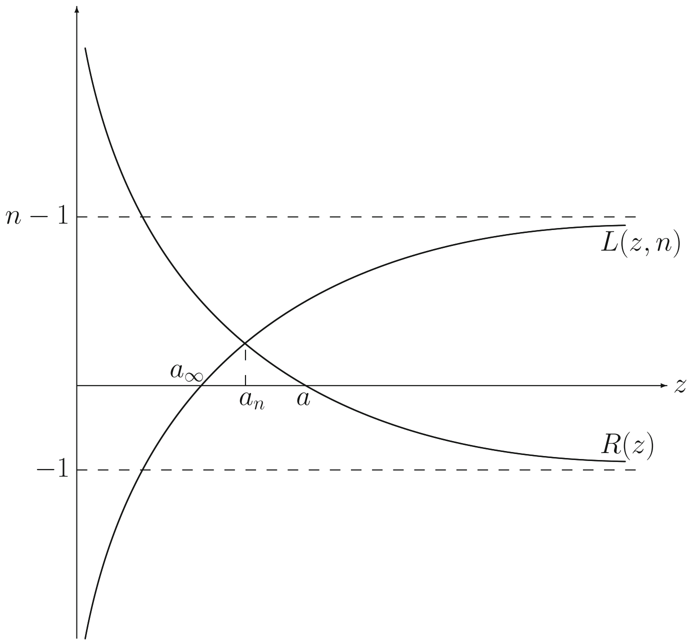

Fig. 1 Illustration of the proof

Together with (14), this implies that

$$\lambda_n = \frac{1}{n-1}\left(1 - \frac{a_n}{a}\right) < \frac{1}{n-1}\left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

This completes the proof of part (a) of the theorem.

Since $L(z, n)$ is strictly increasing with respect to both z and n (as long as $L(z, n)$ is positive) whereas $R(z)$ is independent of n, it follows that the unique value a_n satisfying (17) must be strictly decreasing with respect to n. Together with (16), this implies that λ_n is also strictly decreasing as a function of n.

We have already shown that $a_1 = a$ and $\lambda_1 = \lambda$. Using (15), we have $a_n > \rho^{1/(\eta-1)}$ and so the right-hand side of (17) is bounded above as n approaches $+\infty$. Since a_n is decreasing in n, and a_n is bounded below by $\rho^{1/(\eta-1)}$, the limit $a_\infty := \lim_{n\to+\infty} a_n$ exists. Using (17), we can then infer that

$$1 - \left(\frac{a_\infty^{\eta-1}}{\rho}\right)^{-1/\eta} = 0$$

such that the term in brackets in Eq. (17) vanishes. This shows $a_\infty = \rho^{1/(\eta-1)}$. As a_n has a finite limit and the bracket in (16) approaches 0 as n approaches $+\infty$, it follows that $\lim_{n\to+\infty} \lambda_n = 0$. This completes the proof of part (b).

Consider the equilibrium dynamics generated by the strict MPNE described in Theorem 1, which is given by

$$x_{t+1} = h_n(x_t) := (1 - n\lambda_n) f(x_t). \tag{18}$$

We have the following result.

Theorem 2 *Assume that $a^{1-\eta}\rho < 1$ and $n \in \mathbb{N}$.*

(a) *If $(1 - n\lambda_n)a > 1$, then there exists a unique positive value x_n^* such that $h_n(x_n^*) = x_n^*$. All trajectories of (18) that start from an initial state $x_0 > 0$ converge to x_n^*. The steady-state value x_n^* is a decreasing function of n.*
(b) *If $(1 - n\lambda_n)a \leq 1$, then there does not exist a positive fixed point of the difference Eq. (18). All trajectories of (18) converge to $x = 0$.*

Proof Note that $h_n(0) = 0$ and that $h_n(x)$ is strictly increasing and strictly concave with respect to x. Furthermore, we have

$$h_n'(0) = (1 - n\lambda_n) f'(0) = (1 - n\lambda_n)a$$

and

$$\lim_{x\to+\infty} h_n'(x) = (1 - n\lambda_n) \lim_{x\to+\infty} f'(x) = 0.$$

Except for the monotonicity of x_n^* with respect to n, the theorem follows from these observations. To prove that x_n^* is decreasing in n, it is obviously sufficient to show that $1 - n\lambda_n$ is decreasing with respect to n. From (14), we obtain

$$1 - n\lambda_n = \frac{na_n - a}{a(n-1)}. \tag{19}$$

Furthermore, note that (17) can also be written as

$$\left(\frac{a_n^{\eta-1}}{\rho}\right)^{-1/\eta} = \frac{na_n - a}{a_n(n-1)}.$$

Combining these two equations, it follows that

$$1 - n\lambda_n = \frac{(a_n\rho)^{1/\eta}}{a}.$$

We have shown in the proof of theorem 1 that a_n is decreasing with respect to n. Hence, $1 - n\lambda_n$ must also be decreasing and the proof of the theorem is complete. □

Finally, we show that the resource stock will get extinct whenever the number of agents exceeds a certain threshold value.

Theorem 3 *Assume that* $a^{1-\eta}\rho < 1$. *In the strict MPNE described in Theorem 1(a), extinction occurs if and only if*

$$n \geq \frac{(a-1)\rho}{1-\rho}. \tag{20}$$

Proof If $n = 1$, then (20) is equivalent to $a\rho \leq 1$ and the result follows therefore from Lemma 2. Now assume that $n > 1$. From (19), we know that

$$(1 - n\lambda_n)a = \frac{na_n - a}{n-1}.$$

From Theorem 2, we know that extinction occurs in equilibrium if and only if this expression is less than or equal to 1, which means

$$a_n \leq \tilde{a}_n := \frac{a+n-1}{n}.$$

Using the notation introduced in the proof of theorem 1 and recalling Fig. 1, it is clear that this inequality holds if and only if

$$L(\tilde{a}_n, n) \geq R(\tilde{a}_n).$$

Substituting the definition of $\tilde{a}_n$ into this inequality, we obtain after straightforward rearrangements condition (20). □

The extinction formula (20) forms a strong restriction on the number of agents who can be supported by a common property renewable resource. It is instructive to rewrite this restriction in terms of the intrinsic growth rate of the biomass at a vanishing population, which is given by

$$g = \lim_{x \to 0} \frac{f(x) - x}{x} = f'(0) - 1$$

and the time-preference rate

$$r = \frac{1}{\rho} - 1.$$

Indeed, using these definitions, condition (20) becomes

$$g \leq nr. \tag{21}$$

If n agents with time-preference rates r exploit the resource, then the value nr is the largest lower bound on the intrinsic growth rates of the resource, which prevent extinction. Hence, the value nr can be used as a quantitative measure of the severeness of the tragedy of the commons.

It is interesting to note that the very same condition has also been mentioned by Sorger (1998). There exist, however, crucial differences between Sorger (1998) and the present paper, both in terms of the assumptions and in terms of the results. As for the assumptions, Sorger (1998) studies a continuous-time model with a fixed upper bound on the extraction rate of each player. Moreover, although Sorger (1998) does not specify the functional forms of the natural growth function f or the utility function u (as we do), his assumptions rule out the case considered here, because the utility function is assumed to be finite at 0. In terms of the results, Sorger (1998) constructs infinitely many strict MPNE along which extinction does not occur. This construction is carried out under a condition that is inconsistent with the extinction formula (21).[5] Sorger (1998) also derives a necessary and sufficient condition for most rapid extinction, but this condition is hard to relate to the results of the present paper.

4 Conclusion

The purpose of the present paper was to present an analytically tractable dynamic game describing the exploitation of a renewable resource by non-cooperative players, to derive a precise condition under which extinction occurs in equilibrium, and to demonstrate how duality methods can be applied in order to rigorously prove that a certain solution of the Bellman equation qualifies as the optimal value function. Let us conclude the discussion by two brief remarks.

First, we do not claim that the equilibrium discussed in this paper is unique. Quite often, dynamic games on an infinite time horizon have multiple equilibria even if one restricts attention to Markov-perfect ones. The equilibrium studied above, however, is a natural one as it corresponds to the limit of equilibria in appropriately defined finite-time truncations of the game (see, e.g., Levhari and Mirman (1980) for this argument).

As a second comment, we would like to mention that the results derived in this paper can be generalized to the case in which the natural growth of the resource is subject to stochastic shocks. More specifically, if one replaces Eq. (1) by the stochastic difference equation

$$x_{t+1} = \bar{f}(x_t, a_t) - c_t,$$

where

$$\bar{f}(x, a) = \begin{cases} 0 & \text{if } x = 0, \\ a\left(x^{1-\eta} + b\right)^{1/(1-\eta)} & \text{if } x > 0, \end{cases}$$

and where $(a_t)_{t=0}^{+\infty}$ is a sequence of independent and identically distributed random variables with positive values, then Theorem 1 remains true, provided that the assumption $a^{1-\eta}\rho < 1$ is replaced by $\mathbb{E}(a_t^{1-\eta})\rho < 1$. The generalization of Theorems 2 and

[5] The relevant condition is stated in (4) in Sorger (1998), and its relation to our condition (21) from above is discussed in the middle paragraph on page 86 in Sorger (1998).

3 to the stochastic model is also possible, but one has to be careful about the meaning of extinction in the stochastic framework. For example, if we say that extinction does not occur if and only if there exists a non-trivial invariant distribution of the resource stock, then one can apply the results from Kamihigashi (2007) to show that extinction does not occur in equilibrium if

$$n < \frac{\left(e^{\mathbb{E}\ln a_t} - 1\right)\rho}{1-\rho}.$$

Finally, we would like to point out that extinction in stochastic growth models with a representative agent has also been studied by Kamihigashi (2006) and Mitra and Roy (2006).

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5 Appendix

In this "Appendix," we prove that for every $x_0 \geq 0$, the trajectory generated by (10) and the consumption stream defined by $c_t = \lambda f(x_t)$ form an optimal solution to problem (1)–(3).

Suppose first that $x_0 = 0$. Because of $f(0) = 0$, it follows that every feasible state trajectory satisfies $x_t = 0$ for all $t \in \mathbb{N}_0$. Since this is the only feasible state trajectory, it must be an optimal one.

Now assume that $x_0 > 0$ and denote by $(x_t)_{t=0}^{+\infty}$ the unique trajectory of (10) emanating from x_0. Let $(\tilde{x}_t)_{t=0}^{+\infty}$ be any other feasible state trajectory with the same initial state $\tilde{x}_0 = x_0$. Finally, denote by $c_t = \lambda f(x_t)$ and $\tilde{c}_t = f(\tilde{x}_t) - \tilde{x}_{t+1}$ the corresponding control paths. We need to show that

$$\lim_{T\to+\infty} \sum_{t=0}^{T} \rho^t \left[u(c_t) - u(\tilde{c}_t)\right] \geq 0. \tag{22}$$

We define

$$p_t = u'(c_t) \tag{23}$$

for all $t \in \mathbb{N}_0$. Since $x_t > 0$ and u is strictly increasing and continuously differentiable on $(0, +\infty)$, p_t is well defined for all $t \in \mathbb{N}_0$, and it holds that $p_t > 0$. Since c_t maximizes the right-hand side of the Bellman equation given by $u(c)+\rho W(f(x_t)-c)$ and since $c_t \in (0, f(x_t))$, it follows that

$$p_t = u'(c_t) = \rho W'(f(x_t) - c_t) = \rho W'(x_{t+1}).$$

Furthermore, from the envelope theorem applied to the optimization problem in (6), we obtain

$$W'(x_t) = \rho W'(f(x_t) - c_t) f'(x_t) = \rho W'(x_{t+1}) f'(x_t).$$

Combining the last two equations, we get

$$p_t = \rho p_{t+1} f'(x_{t+1}) \tag{24}$$

for all $t \in \mathbb{N}_0$. Let us define for $(x, c) \in \mathbb{R}^2_+$ and $t \in \mathbb{N}_0$

$$L(x, c, t) = u(c) + p_t[f(x) - c]$$

and note that L is strictly concave in (x, c) for all $t \in \mathbb{N}_0$. Furthermore, it holds that

$$L_x(x_t, c_t, t) = p_t f'(x_t), \tag{25}$$
$$L_c(x_t, c_t, t) = u'(c_t) - p_t = 0, \tag{26}$$

where we have used (23). We have

$$\begin{aligned}
&\sum_{t=0}^{T} \rho^t \left[u(c_t) - u(\tilde{c}_t)\right] \\
&= \sum_{t=0}^{T} \rho^t \{L(x_t, c_t, t) - p_t[f(x_t) - c_t] - L(\tilde{x}_t, \tilde{c}_t, t) + p_t[f(\tilde{x}_t) - \tilde{c}_t]\} \\
&\geq \sum_{t=0}^{T} \rho^t \left[L_x(x_t, c_t, t)(x_t - \tilde{x}_t) + L_c(x_t, c_t, t)(c_t - \tilde{c}_t) - p_t(x_{t+1} - \tilde{x}_{t+1})\right] \\
&= \sum_{t=0}^{T} \rho^t \left[p_t f'(x_t)(x_t - \tilde{x}_t) - p_t(x_{t+1} - \tilde{x}_{t+1})\right] \\
&= p_0 f'(x_0)(x_0 - \tilde{x}_0) + \sum_{t=0}^{T} \rho^t \left[\rho p_{t+1} f'(x_{t+1})(x_{t+1} - \tilde{x}_{t+1}) - p_t(x_{t+1} - \tilde{x}_{t+1})\right] \\
&\quad -\rho^{T+1} p_{T+1} f'(x_{T+1})(x_{T+1} - \tilde{x}_{T+1}) \\
&= \sum_{t=0}^{T} \rho^t \left[p_t(x_{t+1} - \tilde{x}_{t+1}) - p_t(x_{t+1} - \tilde{x}_{t+1})\right] - \rho^T p_T(x_{T+1} - \tilde{x}_{T+1}) \\
&\geq -\rho^T p_T x_{T+1}.
\end{aligned}$$

In the first step, we have used the definition of the function L and, in the second step, the concavity of L in (x, c) and the feasibility of the two solutions. In the third step, we have made use of (25)–(26); in step four, we have rearranged the terms; in step five, we have used $x_0 = \tilde{x}_0$ and (24). The last step follows from $\tilde{x}_{T+1} \geq 0$ and $p_T \geq 0$. From the above result, it follows that (22) is implied by the transversality condition

$$\lim_{T\to+\infty} \rho^T p_T x_{T+1} = 0. \tag{27}$$

We distinguish two cases. First, if $a\rho > 1$, then we know from Lemma 2(a) that $\lim_{T\to+\infty} x_T = x^* > 0$, and therefore, $\lim_{T\to+\infty} p_T = \lim_{T\to+\infty} u'(c_T) = \lim_{T\to+\infty} u'(\lambda f(x_T)) = u'(\lambda f(x^*))$. Since both x_{T+1} and p_T have finite limits and $\rho \in (0, 1)$, it is obvious that (27) holds.

Second, let us assume that $a\rho \leq 1$ holds. We have

$$p_T = u'(c_T) = u'(\lambda f(x_T)) = u'([\lambda/(1-\lambda)]x_{T+1}) = \left(\frac{\lambda}{1-\lambda}\right)^{-\eta} x_{T+1}^{-\eta}$$

and it follows therefore that (27) holds if $\lim_{T\to+\infty} \rho^T x_{T+1}^{1-\eta} = 0$. Since this property has been proven in Lemma 2(b), we have verified the transversality condition (27) also in this case.

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